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## Bond percolation in two-dimensional quasi-lattices

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**Abstract.** A renormalisation group analysis of bond percolation in Penrose lattices (the kite-dart lattice and the rhombi lattice) is presented. The critical percolation probability and the correlation-length exponent are estimated.

### 1. Introduction

Quasi-lattices are ordered structures that do not show any translational order but have a bond-orientational order (Levine and Steinhardt 1984, 1986, Socolar and Steinhardt 1986). Since they are situated just between periodic crystals and glassy systems, various physical properties might be different from those of crystalline or glassy systems. Among others, an important question is whether the quasi-lattice belongs to the universality class of periodic lattice concerning phase transitions. In this paper we investigate the bond percolation problem in two-dimensional quasi-lattices. As a typical example of quasi-lattices, we study two types of the Penrose tiling: the rhombi ( $\mathcal{R}$ ) lattice which is made of thin and thick rhombi with single and double arrows on their edges and the kite-dart ( $\mathcal{KD}$ ) lattice which is made by kite and dart tiles (Penrose 1974, Gardner 1977, de Bruijn 1981: see appendix for typical structures). Besides the edges of tiles we include some diagonals of each tile as bonds. Using a decimation renormalisation group method which utilises the unique geometry of the quasi-lattice, we locate the critical surface for percolation in the parameter space and determine the correlation-length critical exponent.

In § 2, we introduce an approximation scheme for the decimation transformation of a local network which appears in the quasi-lattices and demonstrate its effectiveness for the square lattice. In § 3, we examine the  $\mathcal{KD}$  lattice and the  $\mathcal{R}$  lattice. We derive the renormalisation group equations which relate the  $\mathcal{R}$  lattice to the  $\mathcal{KD}$  lattice and vice versa and give the numerical result. Section 4 gives the concluding remarks and a comparison with a recent computer simulation (Lu and Birman 1987). Typical patterns of the quasi-lattices are included in the appendix.

### 2. Decimation for a local network

We consider a part of the network illustrated in figure 1(a) which consists of five lattice sites and four bonds. In the decimation transformation the central site 0 is

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decimated and the local network figure 1(a) is transformed into figure 1(b) which has four sites and five bonds. By taking the succeeding application into consideration we assume that bonds 0-1, 0-2, 0-3 and 0-4 in figure 1(a) are unblocked with probability  $a, b, a$  and  $c$ , respectively, and bonds 1-2, 2-3, 3-4, 4-1 and 2-4 in figure 1(b) are unbroken with probability  $e, e, f, f$  and  $g$ , respectively. In the decimation transformation, the probabilities  $e, f$  and  $g$  are to be determined as functions of  $a, b$  and  $c$ ,  $e = e(a, b, c)$ ,  $f = f(a, b, c)$  and  $g = g(a, b, c)$ , so that all connections between sites in the cell are faithfully represented. There are four distinct classes of connections between four sites in figure 1(b):

$$1:(1-2), (3-2) \quad 2:(1-4), (2-4) \quad 3:(1-3) \quad 4:(2-4). \quad (2.1)$$

We denote the probability that the pairs of sites are connected by  $\Lambda_1-\Lambda_4$  in figure 1(a) and by  $\Gamma_1-\Gamma_4$  in figure 1(b). It is easy to see that these probabilities are given by

$$\Lambda_1 = ab \quad \Lambda_2 = ac \quad \Lambda_3 = a^2 \quad \Lambda_4 = bc \quad (2.2)$$

and

$$\begin{aligned} \Gamma_1 &= 1 - (1 - e)[1 - ef^2(1 - g) - fg] \\ \Gamma_2 &= 1 - (1 - f)[1 - e^2f(1 - g) - eg] \\ \Gamma_3 &= 1 - (1 - e)(1 - f)[(1 + e)(1 + f) - 2efg] \\ \Gamma_4 &= 1 - (1 - ef)^2(1 - g). \end{aligned} \quad (2.3)$$

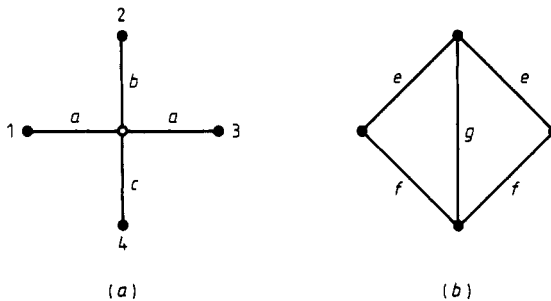
The transformation would be exact if these probabilities are matched for the four classes ( $\Lambda_i = \Gamma_i$  for all  $i = 1-4$ ). Since it is generally impossible to satisfy all four equations we introduce an approximation scheme for the determination of  $e, f$  and  $g$  as functions of  $a, b$  and  $c$ . We use the following three equations for this purpose:

$$\Lambda_1 = \Gamma_1 \quad \Lambda_2 = \Gamma_2 \quad \Lambda_3 + \Lambda_4 = \Gamma_3 + \Gamma_4. \quad (2.4)$$

This set of equations has an advantage because when  $a = b = c$  this reduces to a set of two equations for two unknowns  $e = f$  and  $g$ , which always have solutions in the range 0-1. Equation (2.4) does not always have solutions for arbitrary values of  $(a, b, c)$  between 0 and 1. In the domain of  $(a, b, c)$ , where (2.4) does not have a physical solution, we determine  $(e, f, g)$  by minimising

$$d^2 \equiv 2(\Lambda_1 - \Gamma_1)^2 + 2(\Lambda_2 - \Gamma_2)^2 + 2\left(\frac{\Lambda_3 + \Lambda_4}{2} - \frac{\Gamma_3 + \Gamma_4}{2}\right)^2. \quad (2.5)$$

Note that the weight is given to reflect the number of connections in each class.



**Figure 1.** A local network (a) before and (b) after the decimation. The circles are lattice sites and the full lines are bonds between sites. The open circle in (a) is decimated.

To see the effectiveness of the treatment introduced above, we apply it to the square lattice illustrated in figure 2. We assign the probability of being unblocked,  $k$ , to the nearest-neighbour bonds and  $l$  to the next-nearest-neighbour bonds alternately placed. In each successive transformation, we decimate the sites without the diagonal bonds attached (these are denoted by the open circles in figure 2). To this end the local network (figure 1(b)) is fitted into the sublattice of the diagonal links with the probabilities  $(a, b, c)$  all equal to  $k$ . Let us first take a new bond  $k'$  in the inflated square lattice. Since it is always shared by the two squares, it has the contributions  $f(k, k, k)(=e(k, k, k))$  from both squares. The original diagonal bond  $l$  also contributes to  $k'$ . The new diagonal bond  $l'$  is simply determined by the diagonal of the local network and thus given by  $g(k, k, k)$ . Therefore we obtain the following renormalisation group equations:

$$k' = 1 - (1 - f)^2(1 - l) \quad l' = g. \tag{2.6}$$

The resulting critical surface is shown in figure 3. The critical percolation probability of the square lattice for  $l = 0$  is  $k_c = 0.423$ , to be compared with the exact value  $k_c = 0.5$  (Sykes and Essam 1964). The non-trivial fixed point  $F$  is at  $(k, l) = (0.3329, 0.1868)$ . Near this fixed point, equation (2.6) is linearised as

$$\begin{pmatrix} k' - k_F \\ l' - l_F \end{pmatrix} = \begin{pmatrix} 0.7344 & 0.8204 \\ 0.9710 & 0 \end{pmatrix} \begin{pmatrix} k - k_F \\ l - l_F \end{pmatrix}. \tag{2.7}$$

The largest eigenvalue of the above matrix is  $\lambda_{max} = 1.306$ , which gives the critical exponent

$$\nu_p = \frac{\ln \sqrt{2}}{\ln \lambda_{max}} = 1.2079. \tag{2.8}$$

This should be compared with the exponent  $\frac{4}{3}$  known in the literature (den Nijs 1979, Eshbach *et al* 1981). From these values of  $k_c$  and  $\nu_p$ , we find that our method gives a decent estimate of the characteristics of the critical behaviour.

We should note here that for the square lattice one could use a better scheme in which

$$d'^2 \equiv 2(\Lambda_1 - \Gamma_1)^2 + 2(\Lambda_2 - \Gamma_2)^2 + (\Lambda_3 - \Gamma_3)^2 + (\Lambda_4 - \Gamma_4)^2 \tag{2.9}$$

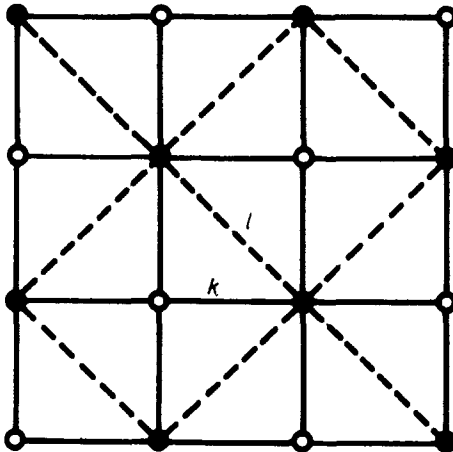
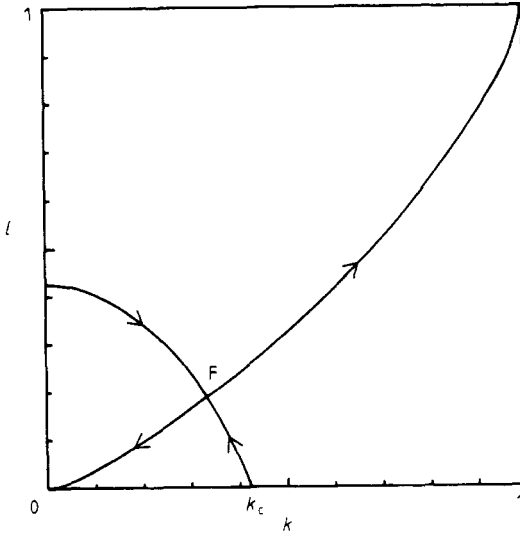


Figure 2. The square lattice with alternative next-nearest-neighbour bonds:  $k$  and  $l$  are the probabilities of nearest- and alternative next-nearest-neighbour bonds being unblocked, respectively. The lattice sites denoted by the open circles are to be decimated.



**Figure 3.** The critical line on the  $(k, l)$  plane for the square lattice. The intersection of the critical line with the axes determines the critical bond percolation probability of the square lattice with only the nearest-neighbour bonds. The non-trivial fixed point is denoted by F.

is minimised to determine  $f$  and  $g$  for all values of  $k$  instead of (2.4). This method yields a non-trivial fixed point at  $(0.4797, 0.1343)$ , which gives the critical exponent  $\nu_p = 1.3000$ , a remarkable agreement with the known exponent. However, it gives the 'trivial fixed point' at  $(1, 0.156)$  instead of  $(1, 1)$  (the other one is at  $(0, 0)$ ). Although this still represents the perfect percolation and thus is conceptually of no inconsistency, when applied to the quasi-lattice, the trivial fixed point does not represent perfect percolation. Thus this method is not acceptable.

### 3. Quasi-lattice

Let us define the bond occupation probabilities  $k_1-k_4$  for the  $\kappa D$  lattice and  $r_1-r_4$  for the  $\mathcal{R}$  lattice as follows:

$$\left\{ \begin{array}{l} k_1: \text{short edge} \\ k_2: \text{long edge} \\ k_3: \text{short 'diagonal' of dart} \\ k_4: \text{short diagonal of kite} \end{array} \right. \quad \left\{ \begin{array}{l} r_1: \text{single-arrowed edge} \\ r_2: \text{double-arrowed edge} \\ r_3: \text{short diagonal of thin rhombus} \\ r_4: \text{long diagonal of thick rhombus.} \end{array} \right. \quad (3.1)$$

The decimation transformation of the quasi-lattices can be done by the successive use of two 'half-inflations' (Aoyama and Odagaki 1986, Henly 1986). One of them (which we denote by an operator  $\mathcal{F}_K^R$ ) 'inflates' a  $\kappa D$  lattice to an  $\mathcal{R}$  lattice and the other ( $\mathcal{F}_R^K$ ) 'inflates' an  $\mathcal{R}$  lattice to a  $\kappa D$  lattice. When these two operations are combined, they give transformations within the same type of quasi-lattices:

$$\mathcal{F}_R^R = \mathcal{F}_K^R \mathcal{F}_R^K \quad \mathcal{F}_K^K = \mathcal{F}_R^K \mathcal{F}_K^R. \quad (3.2)$$

We shall write down the renormalisation group equation for the decimations  $\mathcal{F}_K^R$  and  $\mathcal{F}_R^K$  separately and then combine them according to (3.2) to obtain the transformation for  $\mathcal{F}_K^K$  and  $\mathcal{F}_R^R$ .

$\mathcal{J}_K^R$ : this transformation is illustrated in figure 4. A local network consists of a dart and two halves of kites, which becomes a thick rhombus. The correspondence of the probabilities with the local network of figure 1(a) is  $a = k_1$ ,  $b = k_2$  and  $c = k_3$ . In contrast to the square lattice case, a renormalised bond is not always shared by a unique number of local networks. For example, take a bond  $r'_1$  in the inflated lattice. Besides the contribution of  $k_4$  (which always exists), it has a contribution of  $e$  twice if it is shared by the two local networks, i.e. two thick rhombi (we call this case (1)). Then,  $r'_1$  is given by  $r'_1 = 1 - (1 - e)^2(1 - k_4)$ ,  $e = e(k_1, k_2, k_3)$ . If the  $r'_1$  bond is shared by a thick rhombus and a thin rhombus (case (2)),  $r'_1 = 1 - (1 - e)(1 - k_4)$ . If it is shared by two thin rhombi (case (3)),  $r'_1 = k_4$ . Similarly there are three cases for  $r'_2$ . We determine  $r'_1$  and  $r'_2$  by taking the average over the cases (1)–(3) using the probability of occurrence of each configuration  $P_{r'_i}^{(j)}$  ( $j = 1-3$ ), which has been calculated previously (Aoyama and Odagaki 1986):

$$(P_{r'_1}^{(j)}) = (\phi^3, 2\phi^2, 0) \quad (P_{r'_2}^{(j)}) = (\phi^2, 2\phi^3, \phi^4) \tag{3.3}$$

where  $\phi \equiv (\sqrt{5} - 1)/2$ . Thus, e.g., the probability  $r'_1$  is given by

$$r'_1 = \phi^3[1 - (1 - e)^2(1 - k_4)] + 2\phi^2[1 - (1 - e)(1 - k_4)] + 0k_4. \tag{3.4}$$

The 'diagonal' bonds  $r'_3$  and  $r'_4$  are simply determined by the corresponding bonds, as is evident from the half-inflation rule (figure 4). Thus we arrive at the following renormalisation group equation:

$$\begin{aligned} r'_1 &= 1 - (1 - 2\phi e + \phi^3 e)(1 - k_4) \\ r'_2 &= 1 - (1 - 2\phi f + \phi^2 f^2)(1 - k_2) \\ r'_3 &= k_1 \\ r'_4 &= g. \end{aligned} \tag{3.5}$$

Note that arguments for the functions  $e, f, g$  are  $(k_1, k_2, k_3)$ .

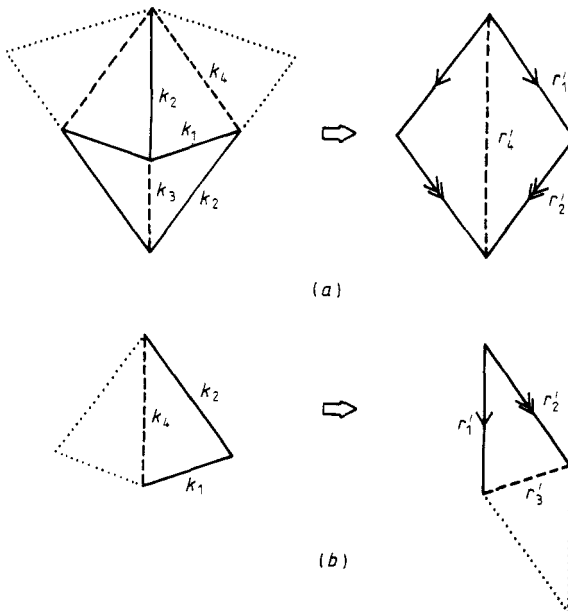


Figure 4. The half-inflation  $\mathcal{J}_K^R$  which transforms the KD lattice to the R lattice.

$\mathcal{J}_R^K$ : as illustrated in figure 5, the relevant local network consists of a thin rhombus and two halves of thick rhombi, which becomes a kite, and  $a = r_1$ ,  $b = r_2$  and  $c = r_3$  in (2.2). For the bonds  $k'_1$  and  $k'_2$ , we need the average procedure using the probabilities  $P_{ki}^{(j)}$  ( $j = 1-3$ ) of the bond being shared by (1) two kites, (2) a kite and a dart and (3) two darts. These are given by (Aoyama and Odagaki 1986)

$$(P_{k'_1}^{(j)}) = (\phi^3, 2\phi^2, 0) \quad (P_{k'_2}^{(j)}) = (2\phi^3, 2\phi^4, \phi^3). \tag{3.6}$$

After a simple analysis, we arrive at the following renormalisation group equations:

$$\begin{aligned} k'_1 &= 1 - (1 - 2\phi f + \phi^3 f)(1 - r_2) \\ k'_2 &= 1 - (1 - 2\phi e + 2\phi^3 e^2)(1 - r_4) \\ k'_3 &= r_1 \\ k'_4 &= g. \end{aligned} \tag{3.7}$$

The arguments for  $e, f, g$  are  $(r_1, r_2, r_3)$ .

We analysed (3.5) and (3.7) numerically and found that they exhibit a typical percolation transition. The critical surface on the  $(r_1, r_2)$  plane is illustrated in figure 6 for various values of  $r_3$  and  $r_4$ . Most features of this critical surface agree with the topological properties of the R lattice, which are obvious from the inspection of a large sample of the lattice. (i) The critical surface passes  $(r_i) = (0, 1, 0, 0)$ . This is in accordance with the fact that the links  $r_2$  cannot form an infinite network by themselves. (ii) For  $r_3 = 1, r_4 = 0$ , it stays within the  $(r_1, r_2)$  plane with  $0 < r_1, r_2 < 1$ , because the links  $r_3$  do not form an infinite network. The critical surface intersects with the line  $r_1 = r_2, r_3 = 0, r_4 = 0$  at  $r_1 = r_2 = 0.410$  which is the critical percolation probability of the R lattice when one type of bond exists only along the edges of the tiles.

The critical surface also passes the point  $(r_i) = (1, 0, 0, 0)$ . This contradicts to the observation that edges with a single arrow seem to form an infinite network by themselves. This is somewhat expected, since the approximation scheme introduced

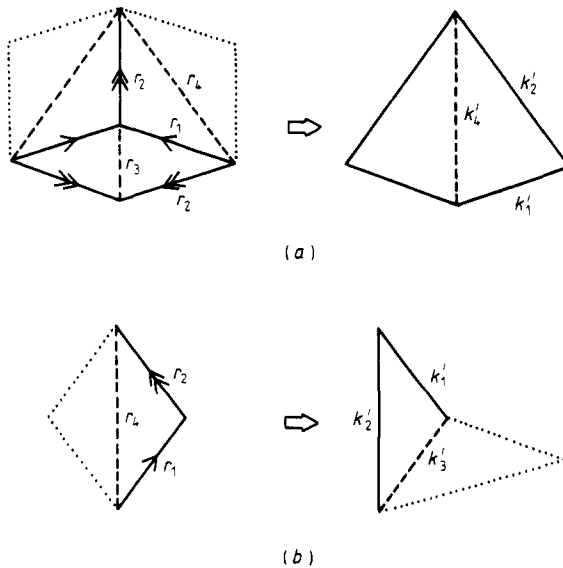
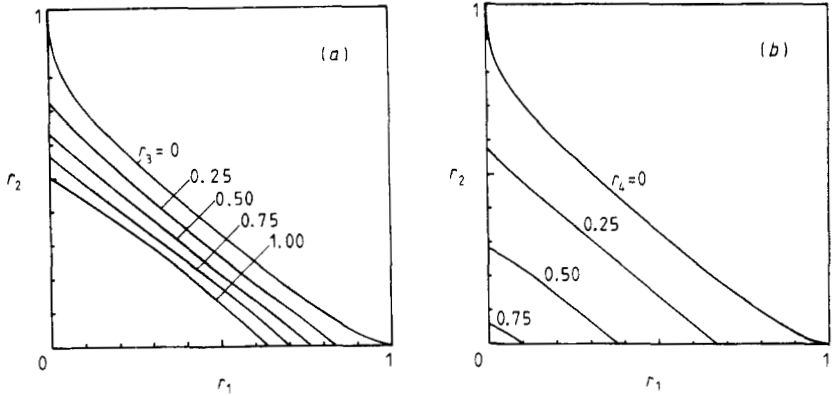


Figure 5. The half-inflation  $\mathcal{J}_R^K$  which transforms the R lattice to the KD lattice.



**Figure 6.** The critical lines on the  $(r_1, r_2)$  plane for the  $\mathbb{R}$  lattice: (a) for  $r_3 = 0, 0.25, 0.5, 0.75$  and  $1$  when  $r_4 = 0$ , (b) for  $r_4 = 0, 0.25, 0.5$  and  $0.75$  when  $r_3 = 0$  (for  $r_4 = 1$ , the critical surface passes the origin).

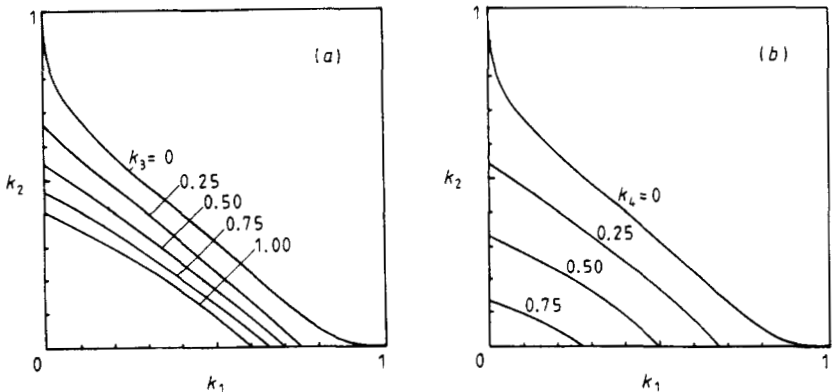
in § 2 always degrades the connection between sites 1 and 3, namely bonds  $r_1$  in this application. The present approximation is not good in the vicinity of  $r_1 = 1$ . (In fact, in this region the ‘distance’  $d$  in (2.5) is forced to be of order of unity.) The result near  $r_1 = r_2$  and the non-trivial fixed points are free from this difficulty.

The non-trivial fixed point for the  $\mathbb{R}$  lattice is at  $(r_i) = (0.2400, 0.3153, 0.3843, 0.1815)$ . The critical index is obtained from the linearised renormalisation group equation near the fixed point. We find the relevant eigenvalue to be  $\lambda_{\max} = 1.4510$ , which yields

$$\nu_p = \frac{\ln \tau}{\ln \lambda_{\max}} = 1.2928. \tag{3.8}$$

(Note that a full inflation changes the linear scale by  $\tau \equiv \phi^{-1}$ .)

The critical lines on the  $(k_1, k_2)$  plane are illustrated in figure 7 for various values of  $k_3$  and  $k_4$ . Their features are quite similar to those of the  $\mathbb{R}$  lattice, since the bonds  $k_1-k_4$  correspond to  $r_1-r_4$ , respectively, in their properties of the connectivity. The non-trivial fixed point for the  $\mathbb{K}\mathbb{D}$  lattice is at  $(k_i) = (0.3843, 0.2438, 0.2400, 0.1578)$ . The



**Figure 7.** The critical lines on the  $(k_1, k_2)$  plane for the  $\mathbb{K}\mathbb{D}$  lattice: (a) for  $k_3 = 0, 0.25, 0.5, 0.75$  and  $1$  when  $k_4 = 0$ , (b) for  $k_4 = 0, 0.25, 0.5$  and  $0.75$  when  $k_3 = 0$  (for  $k_4 = 1$ , the critical surface passes the origin).



critical exponent is the same as (3.8) (for the same reason as in the Ising model (Aoyama and Odagaki 1986)).

#### 4. Concluding remarks

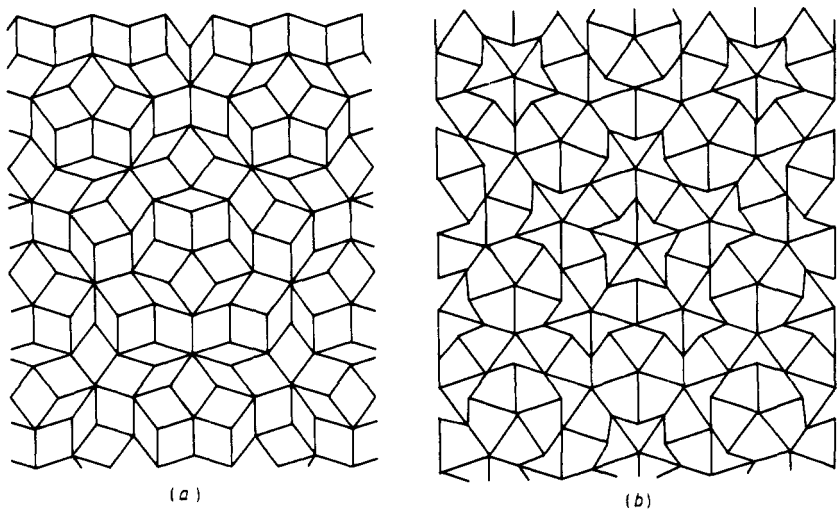
We studied in this paper the bond percolation in two-dimensional quasi-lattices by making use of a real space renormalisation group method. To test the validity of the method, we applied it to the square lattice and found the critical percolation probability  $k_c = 0.423$  and the correlation-length critical exponent  $\nu_p = 1.2079$  which are 16% and 9.5% smaller, respectively, than the values known in the literature. Thus the error in the present methods is estimated to be about 10–15%. These values are, however, better than or comparable to other estimations based on the real space renormalisation group method with similar cell size (Young and Stinchcombe 1975, Stinchcombe and Watson 1976).

The bond percolation in the  $\kappa$  lattice was studied recently by computer simulation by Lu and Birman (1987). When  $r_1 = r_2$ ,  $r_3 = 0$ ,  $r_4 = 0$ , they obtained the critical percolation probability 0.483 and the critical exponent  $\tau = 2.05$  and  $\sigma = 0.39$  which implies  $\nu_p = 1.346$  noting the scaling relation  $\nu_p = (\tau - 1)/2\sigma$  for two dimensions. Our corresponding results,  $r_1 = r_2 = 0.410$  and  $\nu_p = 1.2928$ , agree with these values reasonably well. These estimates for  $\nu_p$  are quite close to the universal value predicted for the two-dimensional periodic lattices  $\frac{4}{3}$  (den Nijs 1979, Eshbach *et al* 1981).

Therefore, our result suggests that the quasi-lattice belongs to the same universality class of two-dimensional periodic lattices.

#### Appendix

Typical patterns of the  $\kappa D$  lattice and the  $\kappa$  lattice are shown in figure 8. These lattices are made of non-periodic tiling using two different tiles as one can easily identify in



**Figure 8.** Typical patterns of two-dimensional quasi-lattices. (a) The  $\kappa$  lattice and (b) the  $\kappa D$  lattice.

the figure: thin and thick rhombi for the  $\mathbb{R}$  lattice and kite and dart tiles for the  $\kappa\mathbb{D}$  lattice. When they are tiled, a matching rule must be satisfied. For example, in the  $\mathbb{R}$  lattice thin and thick rhombi are assigned single and double arrows on their edges (see figures 4 and 5) and the direction and type of arrows must be matched when two tiles are abutted. A similar matching rule is required in the  $\kappa\mathbb{D}$  lattice (see Gardner 1977). The vertices of the tilings form quasi-lattices which have the following properties.

- (i) Existence of minimum and maximum in the nearest-neighbour distance.
- (ii) The bond orientational order.
- (iii) The self-similarity.
- (iv) The quasi-periodicity.

As described in § 2, the  $\mathbb{R}$  and  $\kappa\mathbb{D}$  lattices are transformed into each other with a different length scale by the half-inflation.

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